A new proof of long range scattering for critical nonlinear Schrödinger equations

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Abstract

We present a new proof of global existence and long range scattering, from small initial data, for the one-dimensional cubic gauge invariant nonlinear Schrödinger equation, and for Hartree equations in dimension $n \geq 2$. The proof relies on an analysis in Fourier space, related to the recent works of Germain, Masmoudi and Shatah on space-time resonances. An interesting feature of our approach is that we are able to identify the long range phase correction term through a very natural stationary phase argument.

1 Introduction

The problem of asymptotic behavior of small solutions of nonlinear evolution equations has been extensively treated by many authors in the past fourty years. Almost all of the entire literature on the subject is dedicated to prove that solutions of nonlinear PDEs, evolving from small Cauchy data, behave asymptotically in time like solutions of the associated linear problem. The task of identifying nonlinear global dynamics is extremely challenging. Indeed, there are not many results in the field where a small nonlinear solution is proven to exist globally and behave, for large times, differently than a linear one.

Among the most celebrated results concerning nonlinear Schrödinger equations we want to mention the works of Deift and Zhou[1] (see [5] and references therein) and the work of Hayashi and Naumkin [12]. These latter authors also proved long range scattering results for other dispersive equations such as Benjamin-Ono [13] and KdV [14]. For some results concerning nonlinear asymptotic behavior of solution of wave equations see [18, 1, 17] and references therein.

All of the methods employed to deal with the problems mentioned above do not suggest any unified approach to the question of asymptotic behavior of small solutions when ordinary (linear) scattering fails. Our aim, in this short note, is to present a first application of a new simple method that can be used in situations where the asymptotic behavior of a nonlinear solution differs from the linear one by a phase correction term.

More specifically, we are going to show a new proof of the results contained in [12] about long range scattering for the one dimensional cubic gauge-invariant NLS equation, and for Hartree equations in dimension $n \geq 2$. These equations provide us with an easy setting to expose and implement our method since the necessary computations turn out to be particularly straightdforward. Nevertheless, we believe that our approach is fairly general, and could be used in similar situations. In particular, we think it could be applicable

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[1]The results of these authors, based on the scattering/inverse scattering theory, also apply to large data.

to other scattering critical equations where modified wave operators can be – or have been – constructed with the help of a phase correction term, but asymptotics for solutions of the Cauchy problem are not known yet.

As we will explain below, there are substantial differences between our method and the one used in [12]. In particular, we believe that the main contributions of our approach lie in its simplicity and in the way we explicitly derive the phase correction term. No a priori knowledge of the latter is required, and its derivation is just a mere consequence of a very natural application of the classical stationary phase lemma.

The setting for our proof is inspired from the recent works of Germain, Masmoudi, and Shatah [6, 7, 8] on the analysis of space-time resonances. The very general method developed by these authors, combining earlier ideas about normal forms and commuting vector fields with tools from harmonic analysis, has already proven to be extremely useful in the study of global existence of asymptotically free solutions to nonlinear dispersive equations. In this note we show how a similar approach can be also be applied in the context of long range scattering.

Before formulating the problems and explaining the main ideas of the proof, we give the following **Notations**: For $g \in \mathcal{S}(\mathbb{R}^n)$, we denote its Fourier transform by

$$\widehat{g}(\xi) = \mathcal{F}[g](\xi) = (2\pi)^{-n/2} \int e^{-ix \cdot \xi} g(x) \, dx \,. \tag{1.1}$$

To avoid ambiguities we will sometime use subscripts to indicate the variables with respect to which the Fourier transform is taken. For $m, l \in \mathbb{R}$ we define weighted Sobolev spaces $H^{m,l}$ and $\dot{H}^{m,l}$ by

$$H^{m,l} = \left\{ \varphi \in \mathcal{S}' : \|\varphi\|_{H^{m,l}} := \|\langle x \rangle^l \langle \nabla \rangle^m \varphi\|_{L^2} < \infty \right\}, \tag{1.2}$$

$$\dot{H}^{m,l} = \left\{ \varphi \in \mathcal{S}' : \||x|^l |\nabla|^m \varphi\|_{L^2} < \infty \right\}$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$, $\langle \nabla \rangle := \mathcal{F}^{-1} \langle \xi \rangle \mathcal{F}$ and $|\nabla| := \mathcal{F}^{-1} |\xi| \mathcal{F}$. We denote by $e^{it\Delta/2}$ the fundamental solution of the linear Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$ and define

$$f(t, x) := e^{-it\Delta/2} u(t, x) \tag{1.3}$$

to be the *profile* of a solution u of (1.4).

1.1 The problems

We consider the Cauchy problem for the nonlinear Schrödinger and Hartree equations

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = g(u) \quad , \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \quad , \\ u|_{t=1} = e^{i\Delta/2} u_* \quad , \quad x \in \mathbb{R}^n \quad , \end{cases} \tag{1.4}$$

where the nonlinear term is

$$g(u) = |u|^2 u \quad , \quad x \in \mathbb{R} \tag{1.5}$$

in the NLS case, and

$$g(u) = \left(|x|^{-1} * |u|^2 \right) u \quad , \quad x \in \mathbb{R}^n \quad , \quad n \geq 2 \tag{1.6}$$

in the Hartree case. The nonlinearities (1.5) and (1.6) are critical from the point of view of long time asymptotic behavior, since the L^2–norm of the nonlinear terms, computed on a linear solution, decays like t^{-1}. Moreover, they belong to the case of long range scattering where ordinary wave operators fail to exist [2, 16]. There are many papers treating the problem of long time behaviour for solutions of (1.4) with (1.5)

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or (1.6). Modified wave operators including a certain phase factor were first constructed by Ozawa [20] in the case of (1.5). This result was then extended to other critical NLS equations and Hartree equations by Ginibre and Ozawa [9]. Asymptotic completeness for (1.4)–(1.5) was shown by Naumkin [19], for small data in $H^{1,0} \cap H^{0,1}$. Subsequently, Hayashy and Naumkin [12] proved asymptotic completness for scattering critical NLS equations in dimensions up to 3, and for Hartree equations in dimension $n \geq 2$. They also provided a precise asymptotic formula for solutions of (1.4) with (1.5) or (1.6); see, respectively, (1.14) and (1.21) below.

1.2 Main ideas in the proof

When trying to obtain global existence of small solutions to scattering critical nonlinear dispersive equations, the main issue one often has to face is to establish the sharp (linear) decay of the L_x^∞ norm of solutions. Before the work of Hayashi and Naumkin [12], no author was able to prove that solutions of (1.4) with (1.5), respectively (1.6), satisfy the linear decay estimate $\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}$, respectively $\|u(t)\|_{L^\infty} \lesssim t^{-\frac{n}{2}}$. In order to obtain these bounds, Hayashi and Naumkin introduced an amplitude modulation term – commonly referred to as a "phase correction" – which enabled them to show the desired L^∞ decay. More specifically, in [12] the authors consider the profile f of a solution u, defined as (1.3), and write

$$\partial_t \widehat{f} = e^{it\frac{|\xi|^2}{2}} \mathcal{F}[g(u)]$$
$$= t^{-1} g(\widehat{f}) + R \tag{1.7}$$

where $R = e^{it\frac{|\xi|^2}{2}} \mathcal{F}[g(u)] - t^{-1} g(\widehat{f})$. Subsequently, they perform some manipulations, based on a particular identity for the linear Schrödinger semigroup, in order to show that R decays faster than t^{-1}. The first term in (1.7) is then eliminated by considering a modified profile $\widehat{w}(t,\xi) = \widehat{f}(t,\xi)e^{iB(t,\xi)}$, for a suitable real valued phase function B. The resulting equation yields an a priori L^∞ bound on $\widehat{f}$, which in turn implies the linear decay of solutions.

To understand the difference with our approach, let us briefly describe how we are going to derive the phase correction term. For the sake of discussion we just consider the case of (1.4) with (1.5). We refer to section 2 for a complete proof in this case and to section 3 for the proof in the Hartee case. Following the already cited works of Germain, Masmoudi, and Shatah, we write Duhamel's formula in Fourier space for the profile f:

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - \frac{i}{2\pi} \int_1^t \int e^{is\eta(\xi-\sigma)} \widehat{f}(s,\xi-\eta)\overline{\widehat{f}(s,\sigma-\eta)}\widehat{f}(s,\sigma)\,d\eta d\sigma ds\,. \tag{1.8}$$

We then apply a stationary phase argument to the oscillatory integral with respect to the variables η and σ, obtaining

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i \int_1^t \frac{1}{s} |\widehat{f}(s,\xi)|^2 \widehat{f}(s,\xi)\,ds + \int_1^t R(s,\xi)\,ds\,,$$

which is an integral version of (1.7). Since R can be easily shown to decay faster that s^{-1}, we have explicitly determined, in a natural and direct fashion, the phase correction term which is necessary in the construction of modified wave operators.

Another interesting feature of our approach lies in the way we estimate weighted norms of f. We completely avoid the use of the vector field[2] $J = x - it\nabla$, and the identity $J = e^{it\frac{\Delta}{2}} x e^{-it\frac{\Delta}{2}}$. This vector field approach, despite being customary in the literature, gives, in our opinion, a more restricted perspective

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[2] We use (1.11) and (1.18) to estimate directly $\partial_\xi^k \widehat{f}$ in L^2, which correspond to estimating $J^k u$ in L^2.

on the problem. The Fourier side approach provides, instead, a more general framework, where the particular structure of the gauge invariant non-linearity manifests itself naturally. To see this, let us describe how we establish weighted L^2 bounds on f. Since weights in physical space correspond to derivatives on the Fourier side, we apply ∂_ξ to the integral equation (1.8) associated to (1.4)–(1.5). Then, the main issue is to estimate the contribution of the term where ∂_ξ hits the oscillating phase $e^{is\varphi}$, with $\varphi = \eta(\xi - \sigma)$, because this results in the appearence of a growing factor of s. It is at this stage that the structure of the gauge invariant nonlinearity comes into play. Observe, indeed, that differentiating $e^{is\varphi}$ also introduces a factor $\partial_\xi\varphi = \eta$ that vanishes exactly on the set where $\partial_\sigma\varphi$ vanishes. This suggests that, integrating by parts with respect to σ, one should be able to recover a factor of s^{-1}. In this particularly simple case, one just notices that $\partial_\xi e^{is\eta\cdot(\xi-\sigma)} = -\partial_\sigma e^{is\eta\cdot(\xi-\sigma)}$, so that, after integration by parts, no extra–growth in s is actually present. Notice that the same type of argument would not work for the other cubic nonlineairties u^3, $\bar{u}^3$ or $\bar{u}^2 u$. Although the present example is quite elementary, this type of analysis of "generalized null-structures" constitutes an important part of the methods developed by Germain, Masmoudi and Shatah.

1.3   Reformulation of (1.4) with (1.5) and statement of results

The integral equation associated to (1.4)–(1.5) is

$$u(t) = e^{i(t-1)\partial_{xx}/2}u(1) - i\int_1^t e^{i(t-s)\partial_{xx}/2}|u(s)|^2 u(s)\, ds$$
$$=: e^{it\partial_{xx}/2}u_* + e^{it\partial_{xx}/2}C(u,u,u)(t)\,. \tag{1.9}$$

If f denotes the profile of u defined in (1.3), then

$$f(t) = u_* - i\int_1^t e^{-is\partial_{xx}/2}|u(s)|^2 u(s)\, ds\,.$$

In Fourier space the above equation becomes[3]

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i(2\pi)^{-1}\int_1^t\int e^{is\eta(\xi-\sigma)}\widehat{f}(s,\xi-\eta)\overline{\widehat{f}(s,\sigma-\eta)}\widehat{f}(s,\sigma)\, d\eta d\sigma ds$$
$$=: \widehat{u}_*(\xi) + \widehat{C}(u,u,u)(t)\,, \tag{1.10}$$

since

$$\frac{1}{2}\left[\xi^2 - (\xi-\eta)^2 + (\sigma-\eta)^2 - \sigma^2\right] = \eta(\xi - \sigma)\,.$$

Changing variable in the σ–integral, (1.10) can also be written as

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i(2\pi)^{-1}\int_1^t\int e^{is\eta\sigma}\widehat{f}(s,\xi-\eta)\overline{\widehat{f}(s,\xi-\sigma-\eta)}\widehat{f}(s,\xi-\sigma)\, d\eta d\sigma ds\,. \tag{1.11}$$

We are going to prove the following:

Theorem 1.1. *Let $u_* \in H^{1,0} \cap H^{0,1}$ with $\|u_*\|_{H^{1,0}} + \|u_*\|_{H^{0,1}} \leq \varepsilon$ for some ε small enough. Then there exists a unique global solution $u \in C(\mathbb{R}; H^{1,0}(\mathbb{R}) \cap H^{0,1}(\mathbb{R}))$ of (1.4)–(1.5) such that*

$$\|u(t)\|_{L^\infty} \lesssim \varepsilon(1+|t|)^{-1/2}\,. \tag{1.12}$$

Furthermore, there exists a unique function $W \in L^\infty$ such that for $t \geq 1$

$$\left\|\widehat{f}\exp\left(-i\int_1^t |\widehat{u}(s)|^2\,\frac{ds}{s}\right) - W\right\|_{L^\infty} \lesssim t^{-\delta} \tag{1.13}$$

for some $0 < \delta < \frac{1}{2}$, where f is defined by (1.3).

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[3]With the normalization in (1.1) one has $\mathcal{F}(f*g) = (2\pi)^{n/2}\mathcal{F}(f)\mathcal{F}(g)$ and $\mathcal{F}(fg) = (2\pi)^{-n/2}\mathcal{F}(f)*\mathcal{F}(g)$.

Remarks.

1. Theorem 1.1 corresponds to the $n = 1$ case of theorem 1.1 and (the first part of) theorem 1.2 in [12].

2. It is not hard to see that the function W in theorem 1.1 also belongs to L^2. As a consequence, (1.13) also holds with the L^2 norm replacing the L^∞ norm. We will not provide the details here, since it will be clear how to derive this fact from our computations.

3. The estimate (1.13) is a direct byproduct of the proof of the L^∞ decay estimate (1.12). Furthermore, starting from (1.13), the same arguments of [12, 381-383] can be used to establish the following asymptotic behaviour of solutions of (1.4)–(1.5): there exists a unique function $\Phi \in L^\infty$ such that as $t \to \infty$

$$u(t, x) = \frac{1}{(it)^{1/2}} W(x/t) \exp\left(i \frac{|x|^2}{4t} + i|W(x/t)|^2 \log t + i\Phi(x/t) \right) + O\left(t^{-1/2-\beta} \right) \qquad (1.14)$$

uniformly in $x \in \mathbb{R}$, for some small $\beta > 0$.

In analogy with [12], we define the space

$$X_T := \left\{ u : \|u\|_{X_T} := \|t^{1/2} u\|_{L_T^\infty L_x^\infty} + \|t^{-\alpha} u\|_{L_T^\infty \dot{H}_x^{1,0}} + \|t^{-\alpha} f\|_{L_T^\infty \dot{H}_x^{0,1}} + \|u\|_{L_T^\infty L_x^2} < \infty \right\}$$

where $L_T^\infty := L_t^\infty([0, T])$, f is defined in (1.3) and $\alpha > 0$ is small enough. The global solution of theorem 1.1 will be constructed as a continuation of the local solution provided by the following classical result:

Theorem 1.2 (Local existence [4, 10, 15]). *Assume* $\|u_*\|_{H^{1,0}} + \|u_*\|_{H^{0,1}} \le \varepsilon$, *for some* ε *small enough. Then there exists a finite time* $T > 1$ *and a unique solution* $u \in C([0, T]; H^{0,1}(\mathbb{R}) \cap H^{1,0}(\mathbb{R}))$ *of* (1.4)–(1.5), *such that* $\|u\|_{X_T} \le C\varepsilon$.

The first part of theorem 1.1 will then result as a consequence of

Proposition 1.3. *Let* u *be given as in theorem 1.2, then*

$$\|u\|_{X_T} \le \varepsilon + C\|u\|_{X_T}^3 \,, \qquad (1.15)$$

for some constant C *independent of* T.

The proof of proposition 1.3 is performed in section 2.

1.4 Reformulation of (1.4) with (1.6) and statement of results

Let us consider now the Cauchy problem for the Hartree equation with a Coulomb potential (1.4)–(1.6). The corresponding integral equation is

$$u(t) = e^{i(t-1)\Delta/2} u(1) - i \int_1^\infty e^{i(t-s)\Delta/2} \left(|x|^{-1} * |u(s)|^2 \right) u(s) \, ds$$

$$=: e^{it\Delta/2} u_* + e^{it\Delta/2} C(u, u, u)(t) \qquad (1.16)$$

so that the profile f satisfies

$$\widehat{f}(t, \xi) = \widehat{u}_*(\xi) - i(2\pi)^{-n/2} C_1 \int_1^t \int e^{is\eta \cdot (\xi - \sigma)} |\eta|^{-n+1} \widehat{f}(s, \xi - \eta) \overline{\widehat{f}(s, \sigma - \eta)} \widehat{f}(s, \sigma) \, d\eta d\sigma ds \qquad (1.17)$$

$$= \widehat{u}_*(\xi) + \widehat{C}(u, u, u) \,,$$

since

$$\frac{1}{2}\left[|\xi|^2 - |\xi - \eta|^2 + |\sigma - \eta|^2 - |\sigma|^2\right] = \xi \cdot \eta - \sigma \cdot \eta.$$

Here[4] $C_1 := 2^{n/2-1}\pi^{-1/2}\Gamma((n-1)/2)$. After a change of variables, the integral equation (1.17) can be written as

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i(2\pi)^{-n/2}C_1 \int_1^t \int e^{is\eta\cdot\sigma}|\eta|^{-n+1}\widehat{f}(s,\xi-\eta)\overline{\widehat{f}(s,\xi-\sigma-\eta)}\widehat{f}(s,\xi-\sigma)\,d\eta d\sigma ds. \quad (1.18)$$

We are going to prove the following:

Theorem 1.4. *Let* $m := [n/2] + 1$. *Assume that* $u_* \in H^{m,0} \cap H^{0,m}$ *and* $\|u_*\|_{H^{m,0}} + \|u_*\|_{H^{0,m}} \leq \varepsilon$ *for* ε *small enough. Then there exists a unique global solution* $u \in C(\mathbb{R}; H^{m,0}(\mathbb{R}^n) \cap H^{0,m}(\mathbb{R}^n))$ *of* (1.4)–(1.6), *such that*

$$\|u(t)\|_{L^\infty} \lesssim \varepsilon(1+|t|)^{-n/2}. \quad (1.19)$$

Furthermore, there exists a unique function $W \in L^\infty$ *such that for* $t \geq 1$

$$\left\|\widehat{f}\exp\left(-i\int_1^t\left(|x|^{-1}*|\widehat{u}|^2\right)(s)\frac{ds}{s}\right) - W\right\|_{L^\infty} \lesssim t^{-\delta} \quad (1.20)$$

for some $0 < \delta < \frac{1}{2}$.

Theorem 1.4 above corresponds to Theorem 1.1$'$ and (part of) Theorem 1.2$'$ in [12]. The remarks made after theorem 1.1 apply also to the function W in theorem 1.4 and to the inequality (1.20). Furthermore, the same arguments in [12, 387-388] can again be used to determine the asymptotic behaviour of solution of (1.4)–(1.6), i.e. as $t \to \infty$

$$u(t,x) = \frac{1}{(it)^{n/2}}W(x/t)\exp\left(i\frac{|x|^2}{4t} + i\left(|x|^{-1}*|W|^2\right)(x/t)\log t + i\Phi(x/t)\right) + O\left(t^{-n/2-\alpha}\right) \quad (1.21)$$

for some uniquely determined function $\Phi \in L^\infty$, uniformly for $x \in \mathbb{R}^n$.

For the proof of theorem 1.4 we are going to follow the same strategy adopted in the proof of theorem 1.1. Again in analogy with [12] we define

$$X_T := \left\{u \,:\, \|u\|_{X_T} := \|t^{n/2}u\|_{L^\infty_T L^\infty_x} + \|t^{-\alpha}u\|_{L^\infty_T \dot{H}^{m,0}_x} + \|t^{-\alpha}f\|_{L^\infty_T \dot{H}^{0,m}_x} + \|u\|_{L^\infty_T L^2_x} < \infty\right\}.$$

Global solutions to (1.4)–(1.6) will be constructed as a continuation of local solutions provided by

Theorem 1.5 (Local existence [11, 16, 15]). *Assume* $\|u_*\|_{H^{m,0}} + \|u_*\|_{H^{0,m}} \leq \varepsilon$, *for some* ε *small enough. Then there exists a finite time* $T > 1$ *and a unique solution* $u \in C([0,T]; H^{0,m}(\mathbb{R}^n) \cap H^{m,0}(\mathbb{R}^n))$ *of* (1.4)–(1.6), *such that* $\|u\|_{X_T} \leq C\varepsilon$.

In order to prove the first part of Theorem 1.4, it will then be sufficient to establish the following a priori estimates:

Proposition 1.6. *Let* u *be given as in 1.5. Then, there exists a constant* C, *independent of* T, *such that*

$$\|u\|_{X_T} \leq \varepsilon + C\|u\|_{X_T}^3. \quad (1.22)$$

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[4]We are using the fact that

$$\mathcal{F}\left[|x|^{-a}\right] = 2^{n/2-a}\frac{\Gamma((n-a)/2)}{\Gamma(a/2)}|x|^{a-n} =: C_a|x|^{a-n}.$$

Notice that $C_1 C_{n-1} = 1$.

The proof of proposition 1.6 is performed in section 3.

Before turning to the proofs, we recall some standard properties of the linear Schrödinger semigroup.

Lemma 1.7 ([12]). *The Schrödinger semigroup satisfies the identity*

$$(e^{it\Delta/2}g)(x) = \frac{1}{(it)^{n/2}} e^{i\frac{|x|^2}{2t}} \widehat{g}(x/t) + \frac{1}{t^{n/2+\beta}} O\left(\|g\|_{H^{0,\gamma}}\right)$$

for $x \in \mathbb{R}^n$ and any $\gamma > \frac{n}{2} + 2\beta$. In particular, it follows

$$\|e^{it\Delta/2}g\|_{L^\infty} \lesssim \frac{1}{t^{n/2}} \|\widehat{g}\|_{L^\infty} + \frac{1}{t^{n/2+\beta}} \|g\|_{H^{0,\gamma}}.$$

Furthermore, for any $2 \le p \le \infty$ one has the dispersive estimate:

$$\|e^{it\Delta/2}g\|_{L^p} \lesssim t^{-n(1/2-1/p)} \|g\|_{L^{p'}}. \tag{1.23}$$

Remark. Interpolating between the L_x^2 and L_x^∞ components of the X_T norm, we have[5]

$$\|u\|_{L^{p,q}} \lesssim t^{-n(1/2-1/p)} \|u\|_{X_T}, \tag{1.24}$$

for any $2 < p < \infty$ and $1 \le q \le \infty$.

2 Proof of proposition 1.3

Notice first that a solution to (1.4) with (1.5) or (1.6) enjoys conservation of the L^2 norm.
$\dot{H}^{1,0} \cap \dot{H}^{0,1}$ *estimates*: From (1.9) we have

$$\|\partial_x C\|_{L^2} \lesssim \int_1^t \|\partial_x u(s)\|_{L^2} \|u(s)\|_{L^\infty}^2 \, ds \lesssim \|u\|_{X_T}^3 \int_1^t s^\alpha s^{-1} \, ds \lesssim \|u\|_{X_T}^3 t^\alpha$$

To estimate $\|xC\|_{L^2} = \|\partial_\xi \widehat{C}\|_{L^2}$ we use (1.11). Then

$$\partial_\xi \int_1^t \int e^{is\eta\sigma} \widehat{f}(s, \xi-\eta) \overline{\widehat{f}(s, \xi-\sigma-\eta)} \widehat{f}(s, \xi-\sigma) \, d\eta d\sigma \, ds$$

$$= \int_1^t \int e^{is\eta\sigma} \partial_\xi \widehat{f}(s, \xi-\eta) \overline{\widehat{f}(s, \xi-\sigma-\eta)} \widehat{f}(s, \xi-\sigma) \, d\eta d\sigma \, ds \quad + \quad \text{similar terms}$$

where "similar terms" denotes the terms where the derivative with respect to ξ falls on the Fourier transform of the other profiles. These terms clearly enjoy the same estimates as the first one above. Therefore, redistributing the phases on the three profiles, we see that

$$\|\partial_\xi \widehat{C}\|_{L^2} = \left\| \int_1^t e^{-is\partial_{xx}} \left(e^{is\partial_{xx}/2}(xf) e^{-is\partial_{xx}/2} \bar{f} e^{is\partial_{xx}/2} f \right) \, ds \right\|_{L^2}$$

$$\lesssim \int_1^t \|u\|_{L^\infty}^2 \|xf\|_{L^2} \, ds$$

$$\lesssim \int_1^t s^{-1} s^\alpha \, ds \, \|u\|_{X_T}^3 \lesssim \|u\|_{X_T}^3 t^\alpha.$$

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[5]$L^{p,q}$ denotes the usual Lorentz space [3].

L^∞ *estimate*: As we already pointed out, the advantage of writing (1.4)–(1.5) as an integral equation for the profile, is that one gets a clear understanding of why an asymptotic behavior such as (1.13) – and eventually as (1.14) – occurs. Indeed, following the proof of the classical stationary phase lemma, we use Plancharel's identity in (1.11) to obtain[6]

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i(2\pi)^{-1} \int_1^t \int \mathcal{F}_{\eta,\sigma}\left[e^{is\eta\sigma}\right] \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi)\, d\eta d\sigma ds$$

$$= \widehat{u}_*(\xi) - i(2\pi)^{-1} \int_1^t \int \frac{1}{s} e^{-i\eta\sigma/s} \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi)\, d\eta d\sigma ds$$

where

$$F(s,\eta,\sigma,\xi) := \widehat{f}(s,\xi-\eta)\overline{\widehat{f}(s,\xi-\sigma-\eta)}\widehat{f}(s,\xi-\sigma)\,. \tag{2.1}$$

Then

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i(2\pi)^{-1} \int_1^t \frac{1}{s} \int \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi)\, d\eta d\sigma ds$$

$$- i(2\pi)^{-1} \int_1^t \frac{1}{s} \int \left(e^{-i\eta\sigma/s} - 1\right) \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi)\, d\eta d\sigma ds$$

$$= \widehat{u}_*(\xi) - i \int_1^t \frac{1}{s} |\widehat{f}(s,\xi)|^2 \widehat{f}(s,\xi)\, ds + \int_1^t R(s,\xi)\, ds \tag{2.2}$$

with

$$R(s,\xi) := -\frac{i}{s}(2\pi)^{-1} \int \left(e^{-i\eta\sigma/s} - 1\right) \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi)\, d\eta d\sigma \tag{2.3}$$

We now claim that R satisfies

$$|R(s,\xi)| \lesssim s^{-1-\delta+3\alpha} \|u\|_{X_T}^3 \tag{2.4}$$

for any $3\alpha < \delta < \frac{1}{2}$, uniformly in ξ. We postpone for the moment the proof of this fact and continue the proof of the L^∞ decay estimate. Taking a derivative with respect to time in (2.2) one obtain

$$\partial_t \widehat{f}(t,\xi) = -\frac{i}{t}|\widehat{f}(t,\xi)|^2 \widehat{f}(t,\xi) + R(t,\xi)\,. \tag{2.5}$$

Since R has a better decay in time, it is clear that one should consider

$$\widehat{w}(t,\xi) = \widehat{f}(t,\xi)B(t,\xi) \quad \text{with} \quad B(t,\xi) = \exp\left(i \int_1^t \frac{|\widehat{f}(s,\xi)|^2}{s}\, ds\right), \tag{2.6}$$

so that

$$\partial_t \widehat{w}(t,\xi) = B(t,\xi)R(t,\xi)\,,$$

hence,

$$|\widehat{f}(t,\xi)| = |\widehat{w}(t,\xi)| \leq |\widehat{u}_*(\xi)| + \int_1^t |\partial_t \widehat{w}(s,\xi)|\, ds$$

$$\leq |\widehat{u}_*(\xi)| + \int_1^t |R(s,\xi)|\, ds \lesssim \|u_*\|_{H^{0,1}} + t^{-\delta+3\alpha}\|u\|_{X_T}^3\,.$$

---

[6]We use the identity $\mathcal{F}\left[\exp\left(i\langle x, Qx\rangle/2\right)\right] = e^{i\frac{\pi}{4}\operatorname{sign} Q}|\det Q|^{-1/2} \exp\left(-i\langle\xi, Q^{-1}\xi\rangle/2\right)$, with $Q = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}$.

Using lemma 1.7 with $n = 1$, $3\alpha < \delta$ and $\alpha < \beta < \frac{1}{4}$, one gets

$$\|u\|_{L^\infty} \lesssim \frac{1}{t^{1/2}} \left(\|u_*\|_{H^{0,1}} + t^{-\delta+3\alpha} \|u\|_{X_T}^3 \right) + \frac{1}{t^{1/2+\beta}} \|f\|_{H^{0,1}} \lesssim \frac{1}{t^{1/2}} \left(\|u_*\|_{H^{0,1}} + \|u\|_{X_T}^3 \right).$$

This concludes the proof of (1.15) and implies global existence for solutions of (1.4)–(1.5).

Estimate for the remainder (2.4): From the definition of R we immediately get

$$|R(s,\xi)| \lesssim s^{-1-\delta} \int |\sigma|^\delta |\eta|^\delta \left| \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi) \right| \, d\sigma d\eta$$

for any $0 < \delta < \frac{1}{2}$. To estimate this term we write an explicit expression for $\mathcal{F}_{\eta,\sigma}^{-1}[F]$. Since

$$\mathcal{F}_\eta^{-1}\left[\widehat{f}(\xi - \eta)\right](\eta') = e^{i\eta'\xi} f(-\eta') \quad \text{and} \quad \mathcal{F}_\eta^{-1}\left[\overline{\widehat{f}(\xi - \eta - \sigma)}\right](\eta') = e^{i\eta'(\xi-\sigma)}\overline{f}(\eta'),$$

one has

$$\mathcal{F}_\eta^{-1}\left[\widehat{f}(\xi - \eta)\overline{\widehat{f}(\xi - \eta - \sigma)}\right](\eta') = (2\pi)^{1/2} \mathcal{F}_\eta^{-1}[\widehat{f}(\xi - \eta)] * \mathcal{F}_\eta^{-1}\left[\overline{\widehat{f}(\xi - \eta - \sigma)}\right](\eta')$$

$$= (2\pi)^{1/2} \int e^{i(\eta'-x)\xi} f(x - \eta') \, e^{ix(\xi-\sigma)}\overline{f}(x) \, dx$$

$$= (2\pi)^{1/2} e^{i\xi\eta'} \int e^{-ix\sigma} f(x - \eta')\overline{f}(x) \, dx$$

Moreover, since $\mathcal{F}_\sigma^{-1}[e^{-ix\sigma}\,\widehat{f}(\xi - \sigma)](\sigma') = e^{i\xi(\sigma'+x)} f(x - \sigma')$ we have

$$\mathcal{F}_{\eta,\sigma}^{-1}[F] = \mathcal{F}_{\eta,\sigma}^{-1}\left[\widehat{f}(\xi - \eta)\overline{\widehat{f}(\xi - \sigma - \eta)}\widehat{f}(\xi - \sigma)\right]$$

$$= (2\pi)^{1/2} e^{i\xi\eta'} \mathcal{F}_\sigma^{-1}\left[\int e^{-ix\sigma} f(x - \eta')\overline{f}(x) \, dx \, \widehat{f}(\xi - \sigma)\right]$$

$$= (2\pi)^{1/2} e^{i\xi(\eta'+\sigma')} \int e^{-ix\xi} f(x - \eta')\overline{f}(x)f(x - \sigma') \, dx. \tag{2.7}$$

Therefore,

$$|R(s,\xi)| \lesssim s^{-1-\delta} \int |\eta|^\delta |\sigma|^\delta |f(s, x - \eta)||\overline{f}(s,x)||f(s, x - \sigma)| \, dx \, d\eta \, d\sigma.$$

Using $\delta < \frac{1}{4}$ and $|\eta|^\delta|\sigma|^\delta \lesssim \left(|x - \eta|^\delta + |x|^\delta \right)\left(|x - \sigma|^\delta + |x|^\delta \right)$, we can conclude that

$$|R(s,\xi)| \lesssim s^{-1-\delta} \left\| \langle x \rangle^{2\delta} f \right\|_{L^1}^2 \|f\|_{L^1} \lesssim s^{-1-\delta} \|f\|_{H^{0,1}}^3 \lesssim s^{-1-\delta+3\alpha} \|u\|_{X_T}^3 \quad \square$$

3 Proof of proposition 1.6

$\dot{H}^{m,0}$ *estimates:* For any $|k| = m$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ we have

$$\|\partial^k C\|_{L^2} \lesssim \int_1^t \left\| |x|^{-1} * |u|^2 \right\|_{L^\infty} \|\partial^k u\|_{L^2} \, ds \tag{3.1}$$

$$+ \int_1^t \left\| |x|^{-1} * \partial^k |u|^2 \right\|_{L^q} \|u\|_{L^p} \, ds. \tag{3.2}$$

Using the endpoint of the Hardy–Littlewood–Sobolev inequality and (1.24), we get

$$\int_1^t \left\| |x|^{-1} * |u|^2 \right\|_{L^\infty} \|\partial^k u\|_{L^2}\, ds \lesssim \int_1^t \|u\|^2_{L^{\frac{2n}{n-1},2}} \|u\|_{X_T} s^\alpha\, ds$$

$$\lesssim \|u\|^3_{X_T} \int_1^t s^{-1} s^\alpha\, ds \lesssim \|u\|^3_{X_T} t^\alpha.$$

To estimate (3.2) we choose q_1 and q_2 satisfying $\frac{1}{q} = \frac{1}{q_1} - \frac{n-1}{n}$ and $\frac{1}{2} + \frac{1}{q_2} = \frac{1}{q_1}$, so that

$$\left\| |x|^{-1} * \partial^k |u|^2 \right\|_{L^q} \lesssim \|\partial^k |u|^2\|_{L^{q_1}} \lesssim \|u\|_{L^{q_2}} \|\partial^k u\|_{L^2}.$$

Using again (1.24) we obatin

$$\int_1^t \left\| |x|^{-1} * \partial^k |u|^2 \right\|_{L^q} \|u\|_{L^p}\, ds \lesssim \int_1^t \|u\|_{L^{q_2}} \|\partial^k u\|_{L^2} \|u\|_{L^p}\, ds$$

$$\lesssim \|u\|^3_{X_T} \int_1^t s^\alpha s^{-n(1-1/p-1/q_2)}\, ds \lesssim \|u\|^3_{X_T} t^\alpha.$$

$\dot{H}^{0,m}$ *estimate in dimension* $n \geq 3$: Let k be any multi–index with $|k| = m$. Applying ∂_ξ^k to $\widehat{C}$ produces a linear combination of terms of the form

$$\int_1^t \int e^{is\eta\cdot\sigma} |\eta|^{-n+1} \partial_\xi^k \widehat{f}(s, \xi - \eta) \left[\overline{\widehat{f}(s, \xi - \eta - \sigma)} \widehat{f}(s, \xi - \sigma) \right] d\eta d\sigma\, ds \tag{3.3}$$

$$\int_1^t \int e^{is\eta\cdot\sigma} |\eta|^{-n+1} \partial_\xi^{k-j} \widehat{f}(s, \xi - \eta)\, \partial_\xi^j \left[\overline{\widehat{f}(s, \xi - \eta - \sigma)} \widehat{f}(s, \xi - \sigma) \right] d\eta d\sigma\, ds \tag{3.4}$$

for $0 \leq |j| \leq m$. Since

$$\partial_\xi^j \left[\overline{\widehat{f}(s, \xi - \eta - \sigma)} \widehat{f}(s, \xi - \sigma) \right] = (-1)^{|j|} \partial_\sigma^j \left[\overline{\widehat{f}(s, \xi - \eta - \sigma)} \widehat{f}(s, \xi - \sigma) \right] \tag{3.5}$$

we can integrate by parts, with respect to σ, transforming (3.4) into a combination of terms of the form

$$\int_1^t \int e^{is\eta\cdot\sigma} (s\eta)^j |\eta|^{-n+1} \partial_\xi^{k-j} \widehat{f}(s, \xi - \eta)\, \overline{\widehat{f}(s, \xi - \eta - \sigma)} \widehat{f}(s, \xi - \sigma)\, d\eta d\sigma\, ds \tag{3.6}$$

Using again the Hardy–Littlewwod–Sobolev inequality, we can estimate

$$\|(3.3)\|_{L^2} \lesssim \int_1^t \left\| |x|^{-1} * |u(s)|^2 \right\|_{L^\infty} \|e^{is\Delta/2} x^k f(s)\|_{L^2}\, ds$$

$$\lesssim \int_1^t \|u(s)\|^2_{L^{\frac{2n}{n-1},2}} s^\alpha \|u\|_{X_T}\, ds$$

$$\lesssim \|u\|^3_{X_T} \int_1^t s^{-1} s^\alpha\, ds \lesssim t^\alpha \|u\|^3_{X_T}.$$

To estimate (3.6) we need the following:

Lemma 3.1. *Let* $1 \leq j \leq m$ *and let* $2 < p < 2n/(n - 2j)$, *then*

$$\|e^{is\Delta/2} |x|^{m-j} f\|_{L^p} \lesssim s^{-n(\frac{1}{2} - \frac{1}{p})} \left(\|f\|_{L^2} + \||x|^m f\|_{L^2} \right).$$

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Proof. Since $p > 2$, we have

$$\|e^{is\Delta/2}|x|^{m-j}f\|_{L^p} \lesssim s^{-n(\frac{1}{2}-\frac{1}{p})}\||x|^{m-j}f\|_{L^{p'}}.$$

Using the upperbound on p, we can estimate

$$
\begin{aligned}
\||x|^{m-j}f\|_{L^{p'}} &\leq \|\chi_{\{|x|\leq 1\}}|x|^{m-j}f\|_{L^{p'}} + \|\chi_{\{|x|>1\}}|x|^{m-j}f\|_{L^{p'}} \\
&\leq \|\chi_{\{|x|\leq 1\}}f\|_{L^{p'}} + \|\chi_{\{|x|>1\}}|x|^{-j}\|_{L^{\frac{2p}{p-2}}}\||x|^{m}f\|_{L^2} \\
&\lesssim \|f\|_{L^2} + \||x|^{m}f\|_{L^2} \quad \square
\end{aligned}
$$

Now we choose p, q and $q_1 \geq 1$ satisfying the following relations:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \quad , \quad 2 < p < \frac{2n}{n-2j} \quad , \quad \frac{1}{q_1} = \frac{1}{q} - \frac{|j|-n+1}{n}.$$

Then, from lemma 3.1 and Sobolev's embedding, we obtain

$$
\begin{aligned}
\|(3.6)\|_{L^2} &\lesssim \int_1^t s^{|j|}\left\|\left(\partial^j|x|^{-1}*|u(s)|^2\right)e^{is\Delta/2}x^{k-j}f(s)\right\|_{L^2} ds \\
&\lesssim \int_1^t s^{|j|}\left\||\nabla|^{|j|-n+1}|u(s)|^2\right\|_{L^q}\|e^{is\Delta/2}x^{k-j}f(s)\|_{L^p} ds. \\
&\lesssim \int_1^t s^{|j|}\|u(s)\|_{L^{2q_1}}^2 s^{-n(\frac{1}{2}-\frac{1}{p})}s^{\alpha}\|u\|_{X_T} ds \\
&\lesssim \|u\|_{X_T}^3 \int_1^t s^{|j|} s^{-n(1-\frac{1}{q_1})}s^{-n(\frac{1}{2}-\frac{1}{p})}s^{\alpha} ds \\
&= \int_1^t s^{-1}s^{\alpha} ds\|u\|_{X_T}^3 \lesssim t^{\alpha}\|u\|_{X_T}^3
\end{aligned}
$$

$\dot{H}^{0,2}$ *estimate in dimension* $n=2$: For any $k \in \mathbb{N}^2$ with $|k| = 2$, $\partial_{\xi}^k\widehat{C}$ is a combination of terms of the form

$$\int_1^t \int e^{is\eta\cdot\sigma}|\eta|^{-1}\partial_{\xi}^2\widehat{f}(s,\xi-\eta)\,\overline{\widehat{f}(s,\xi-\eta-\sigma)}\widehat{f}(s,\xi-\sigma)\,ds\,d\eta\,d\sigma \tag{3.7}$$

$$\int_1^t \int e^{is\eta\cdot\sigma}|\eta|^{-1}\partial_{\xi}\widehat{f}(s,\xi-\eta)\,\partial_{\xi}\{\overline{\widehat{f}(s,\xi-\eta-\sigma)}\widehat{f}(s,\xi-\sigma)\}\,ds\,d\eta\,d\sigma \tag{3.8}$$

$$\int_1^t \int e^{is\eta\cdot\sigma}|\eta|^{-1}\widehat{f}(s,\xi-\eta)\,\partial_{\xi}^2\{\overline{\widehat{f}(s,\xi-\eta-\sigma)}\widehat{f}(s,\xi-\sigma)\}\,ds\,d\eta\,d\sigma. \tag{3.9}$$

The term in (3.7) can be estimated in the same way as (3.3), so we skip it. Using (3.5) again, we can integrate by parts transforming (3.8) and (3.9) in terms of the form

$$\int_1^t \int se^{is\eta\cdot\sigma}\frac{\eta}{|\eta|}\partial_{\xi}\widehat{f}(s,\xi-\eta)\,\{\overline{\widehat{f}(s,\xi-\eta-\sigma)}\widehat{f}(s,\xi-\sigma)\}\,d\eta\,d\sigma\,ds \tag{3.10}$$

$$\int_1^t \int se^{is\eta\cdot\sigma}\frac{\eta}{|\eta|}\widehat{f}(s,\xi-\eta)\,\partial_{\xi}\overline{\widehat{f}(s,\xi-\eta-\sigma)}\widehat{f}(s,\xi-\sigma)\,d\eta\,d\sigma\,ds \tag{3.11}$$

$+$ similar term.

Denoting by $\mathcal{R}$ the Riesz–transform, i.e. $\mathcal{R}f := \mathcal{F}^{-1}(\frac{\eta}{|\eta|}\mathcal{F}f)$, we have

$$\|(3.10)\|_{L^2} = \left\| \int_1^t s \left(\mathcal{R}|u|^2 \right) \left(e^{-is\Delta/2} x f \right) \right\|_{L^2} ds$$

$$\lesssim \int_1^t s \, \||u|^2\|_{L^4} \, s^{-1/2} \|xf\|_{L^{4/3}} \, ds$$

$$\lesssim \int_1^t s^{-1} \|u\|_{X_T}^2 \|\langle x \rangle^2 f\|_{L^2} \, ds \lesssim t^\alpha \|u\|_{X_T}^3 \, .$$

For (3.11) we get

$$\|(3.11)\|_{L^2} = \left\| \int_1^t s\mathcal{R} \left(\bar{u} e^{-is\Delta/2} (xf) \right) e^{-is\Delta/2} f \right\|_{L^2} ds$$

$$\lesssim \int_1^t s \, \|\bar{u} e^{-is\Delta/2}(xf)\|_{L^2} \|e^{-is\Delta/2} f\|_{L^\infty} \, ds$$

$$\lesssim \int_1^t s^{-1} \|u\|_{X_T}^2 \|xf\|_{L^2} \, ds \lesssim t^\alpha \|u\|_{X_T}^3 \, .$$

L^∞ *decay estimate*: We proceed as in the NLS case applying Plancherel's identity to (1.18). Since

$$\mathcal{F}_\eta \left[e^{is\eta \cdot \sigma} |\eta|^{-n+1} \right] (\eta') = \mathcal{F}_\eta \left[|\eta|^{-n+1} \right] (\eta' - s\sigma) = C_{n-1} |\eta' - s\sigma|^{-1}$$

we have

$$\mathcal{F}_{\eta,\sigma} \left[e^{is\eta \cdot \sigma} |\eta|^{-n+1} \right] (\eta', \sigma') = C_{n-1} \mathcal{F}_\sigma \left[|\eta' - s\sigma|^{-1} \right] (\sigma') = s^{-1} e^{i\eta' \cdot \sigma'/s} |\sigma'|^{-n+1} \, .$$

Therefore, we can write

$$\widehat{C}(u,u,u) = -i(2\pi)^{-n/2} C_1 \int_1^t \int e^{is\eta \cdot \sigma} |\eta|^{-n+1} F(s,\xi,\eta,\sigma) \, d\eta d\sigma$$

$$= -i(2\pi)^{-n/2} C_1 \int_1^t \int \frac{1}{s} e^{i\eta \cdot \sigma/s} |\sigma|^{-n+1} \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\xi,\eta,\sigma) \, d\eta d\sigma$$

$$=: \int_1^t I_0(s,\xi) + \int_1^t R(s,\xi),$$

with

$$I_0(s,\xi) := -\frac{i}{s}(2\pi)^{-n/2} C_1 \int |\sigma|^{-n+1} \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\xi,\eta,\sigma) \, d\eta d\sigma \, ,$$

$$R(s,\xi) := -\frac{i}{s}(2\pi)^{-n/2} C_1 \int \left(e^{i\eta \cdot \sigma/s} - 1 \right) |\sigma|^{-n+1} \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\xi,\eta,\sigma) \, d\eta d\sigma \qquad (3.12)$$

and F given by (2.1). Next, we observe that

$$I_0(s,\xi) = -\frac{i}{s}(2\pi)^{-n/2} C_1 \int \mathcal{F}_\sigma[|\sigma|^{-n+1}] \mathcal{F}_\eta^{-1}[F](s,\xi,\eta,\sigma) \, d\eta \, d\sigma$$

$$= -\frac{i}{s}(2\pi)^{-n/2} \int |\sigma|^{-1} \mathcal{F}_\eta^{-1}[F](s,\xi,\eta,\sigma) \, d\eta \, d\sigma$$

$$= -\frac{i}{s} \int |\sigma|^{-1} F(s,\xi,0,\sigma) \, d\sigma$$

$$= -\frac{i}{s} \int |\sigma|^{-1} |\widehat{f}(s,\xi-\sigma)|^2 \, d\sigma \, \widehat{f}(s,\xi)$$

$$= -\frac{i}{s} \left(|x|^{-1} * |\widehat{f}(s)|^2 \right) (\xi) \, \widehat{f}(s,\xi) \, .$$

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Therefore,

$$\widehat{f}(t,\xi) = \widehat{u}_*(\xi) - i \int_1^t \frac{1}{s} \left(|x|^{-1} * |\widehat{f}(s)|^2 \right)(\xi)\widehat{f}(s,\xi)\,ds + \int_1^t R(s,\xi)\,ds \tag{3.13}$$

with R given by (3.12). We now claim that R satisfies

$$|R(s,\xi)| \leq s^{-1-\delta+3\alpha}\|u\|_{X_T}^3 \tag{3.14}$$

for any $3\alpha < \delta < \frac{1}{2}$, $\xi \in \mathbb{R}^n$. Assuming for the moment (3.14), we take a derivative with respect to time in (2.2) and obtain

$$\partial_t \widehat{f}(t,\xi) = -\frac{i}{t}\left(|x|^{-1} * |\widehat{f}(t)|^2 \right)\widehat{f}(t,\xi) + R(t,\xi). \tag{3.15}$$

Since R has a better decay in time we eliminate the middle term in (2.5) by considering a modified profile

$$\widehat{w}(t,\xi) = \widehat{f}(t,\xi)B(t,\xi) \quad \text{with} \quad B(t,\xi) = \exp\left(i\int_1^t |x|^{-1} * |\widehat{f}(s)|^2\,ds \right). \tag{3.16}$$

Then $\partial_t \widehat{w}(t,\xi) = B(t,\xi)R(t,\xi)$, whence

$$|\widehat{f}(t,\xi)| = |\widehat{w}(t,\xi)| \leq |\widehat{u}_*(\xi)| + \int_1^t |R(s,\xi)|\,ds \lesssim \|u_*\|_{H^{0,m}} + t^{-\delta+3\alpha}\|u\|_{X_T}^3.$$

Using lemma 1.7, we finally obtain

$$\|u\|_{L^\infty} \lesssim \frac{1}{t^{n/2}}\left(\|u_*\|_{H^{0,m}} + t^{-\delta+3\alpha}\|u\|_{X_T}^3 \right) + \frac{1}{t^{n/2+\beta}}\|f\|_{H^{0,m}} \lesssim \frac{1}{t^{n/2}}\left(\|u_*\|_{H^{0,m}} + \|u\|_{X_T}^3 \right).$$

This concludes the proof of global existence of solutions of (1.4)–(1.6), provided we show (3.14).

Remainder estimate (3.14): From the definition of R we immediately get

$$|R(s,\xi)| \lesssim s^{-1-\delta} \int |\eta|^\delta |\sigma|^{-n+1+\delta} \left| \mathcal{F}_{\eta,\sigma}^{-1}[F](s,\eta,\sigma,\xi) \right|\,d\sigma d\eta$$

for any $0 < \delta < \frac{1}{2}$. Now we use the explicit expression for $\mathcal{F}_{\eta,\sigma}^{-1}[F](s,\xi,\eta,\sigma)$, derived in (2.7), to obtain

$$|R(s,\xi)| \lesssim s^{-1-\delta} \int |\eta|^\delta |\sigma|^{-n+1+\delta} |f(x-\eta)||\overline{f}(x)||f(x-\sigma)|\,dx\,d\eta\,d\sigma.$$

Since $\delta < \frac{1}{2}$, the integral with respect to σ can be directly estimated by

$$\int |\sigma|^{-n+1+\delta}|f(x-\sigma)|\,d\sigma \lesssim \|f\|_{L^\infty} + \|f\|_{H^{0,m}} \lesssim \|u\|_{X_T}s^\alpha.$$

Using $|\eta|^\delta \lesssim |\eta+x|^\delta + |x|^\delta$ we conclude

$$|R(s,\xi)| \lesssim s^{-1-\delta+\alpha} \||x|^\delta f\|_{L^1}\|f\|_{L^1}\|u\|_{X_T} \lesssim s^{-1-\delta+3\alpha}\|u\|_{X_T}^3 \quad \square$$

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